Student Number:

Cheltenham Girls High School

2023 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

Assessment Task 4 – Trial HSC

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black pen only
- Calculators approved by NESA may be used
- A NESA reference sheet is provided at the back of this paper
- For all questions in Section II, show relevant mathematical reasoning and/or calculations to obtain maximum marks.

Total marks: 100

Section I – 10 marks (pages 2-7)

- Attempt Questions 1–10
- Allow about 15 minutes for this section
- Use the multiple choice answer sheet attached to answer Questions 1–10

Section II – 90 marks (pages 8-14)

- Attempt Questions 11-16
- Allow about 2 hours and 45 minutes for this section
- Use separate writing booklets to answer each of the Questions 11 to 16.

Multiple Choice Q1-10	Question 11	Question 12	Question 13	Question 14	Question 15	Question 16	Total
/10	/15	/15	/15	/15	/15	/15	/100

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Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

(USE THE MULTIPLE CHOICE ANSWER SHEET PROVIDED for **Questions 1 – 10**)

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1. If $z = 3 - i$, then the value of $3i - \overline{z}$ is:

(A) $-3 - 2i$

(B) $-3 + 2i$

(C) $3 - 2i$

(D) $3 + 2i$

2. Which of the following represents a sphere with centre $(1, 2, -2)$ and radius 3 units?

(A) $x^2 - 2x + y^2 - 4y + z^2 + 4z = 0$

(B) $x^2 + 2x + y^2 + 4y + z^2 - 4z = 0$

(C) $\left|\begin{pmatrix} x+1 \\ y+2 \\ z-2 \end{pmatrix}\right| = 3$

(D) $\left|\begin{pmatrix} x-1 \\ y-2 \\ z+2 \end{pmatrix}\right| = 9$

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3. The converse of the statement "If x is odd, then y is even" is:

(A) "If y is odd, then x is even".

(B) "If y is even, then x is odd".

(C) "If y is even, then x is even".

(D) "If y is odd, then x is odd".

4. Consider the vectors $\underset{\sim}{a} = \begin{pmatrix} 1.2 \\ 0 \\ k \end{pmatrix}$ and $\underset{\sim}{b} = \begin{pmatrix} 6 \\ 0 \\ -10 \end{pmatrix}$.

Which value of k will make the vectors perpendicular to each other?

(A) $k = -2$

(B) $k = 2$

(C) $k = -\frac{18}{25}$

(D) $k = \frac{18}{25}$

5. Let $\omega = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$. Which of the following polynomials has $\omega, \omega^3, \omega^7$ and ω^9 as its zeros?

(A) $z^4 + z^3 + z^2 + z + 1$

(B) $z^4 + z^3 - z^2 - z + 1$

(C) $z^4 - z^3 - z^2 + z + 1$

(D) $z^4 - z^3 + z^2 - z + 1$

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6. Which of the following is a reduction formula for the integral $I_n = \int x^n e^{2x} dx$?

(A) $2x^n e^{2x} - 2nI_{n-1}$

(B) $2x^n e^{2x} - nI_{n-1}$

(C) $\frac{1}{2}x^n e^{2x} - \frac{1}{2}nI_{n-1}$

(D) $\frac{1}{2}x^n e^{2x} - nI_{n-1}$

7. Which of the following are solutions to $x^2 - 4(1+i)x + 10i = 0$?

(A) $x = 3 + i$ or $x = 1 + 3i$

(B) $x = 3 - i$ or $x = 1 - 3i$

(C) $x = 1 - i$ or $x = -1 + i$

(D) $x = 1 + i$ or $x = -1 - i$

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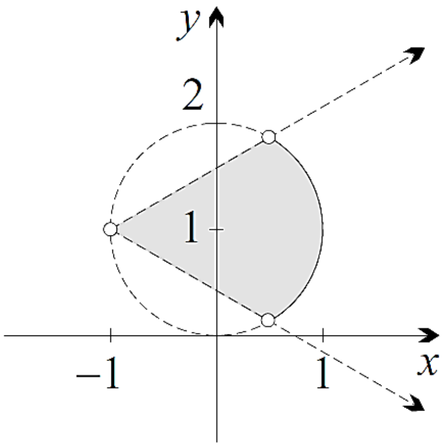
8. The shaded region below is constructed by taking the intersection of two other regions.



NOT TO SCALE

Which of the following best represents the two possible regions?

(A) $|z-1| < 1$ and $-\frac{\pi}{6} \leq \arg(z-1+i) \leq \frac{\pi}{6}$

(B) $|z-i| < 1$ and $-\frac{\pi}{6} \leq \arg(z+1-i) \leq \frac{\pi}{6}$

(C) $|z-1| \leq 1$ and $-\frac{\pi}{6} < \arg(z-1+i) < \frac{\pi}{6}$

(D) $|z-i| \leq 1$ and $-\frac{\pi}{6} < \arg(z+1-i) < \frac{\pi}{6}$

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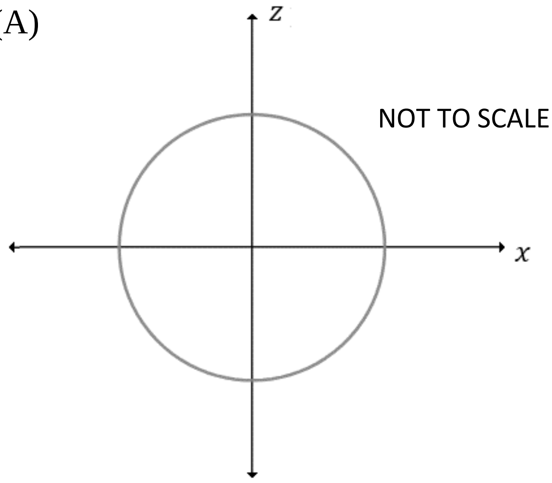
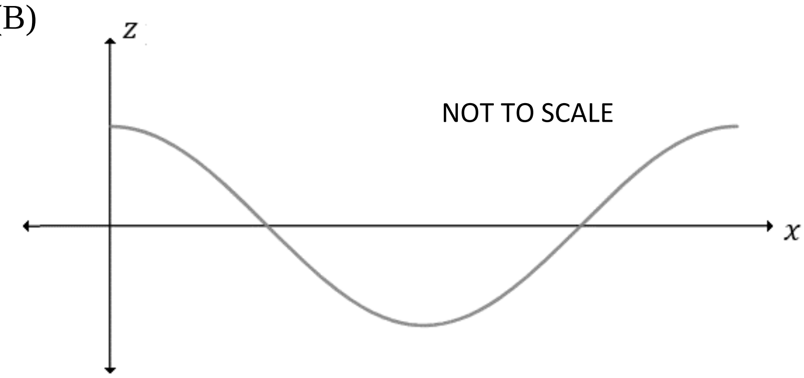
9. Consider the following parametric curve given by $\underset{\sim}{r} = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$ from $0 \leq t \leq 2\pi$.

Which of the following best represents the xz projection of $\underset{\sim}{r}$?

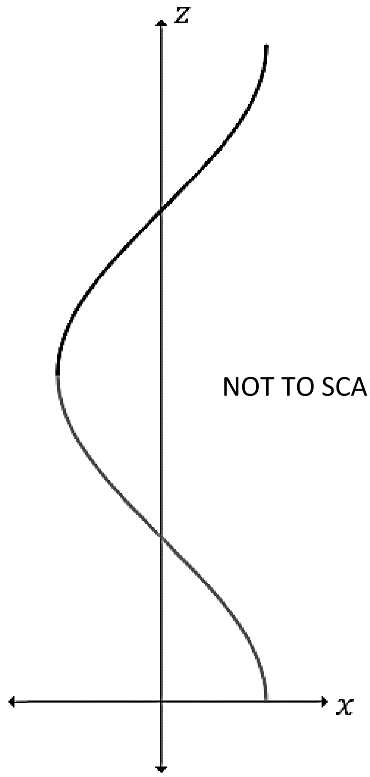
(A)



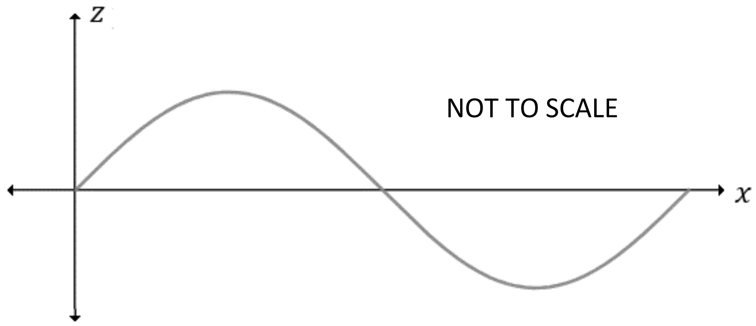
(B)



(C)



(D)



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10. The following induction proof to show that $5^n + 2 \times 11^n$ is a multiple of 3 for all positive integers n, contains an algebraic error.

In which section does the error occur?

(A) Show true for $n = 1$.
$5^1 + 2 \times 11^1 = 27$, which is a multiple of 3.
$\therefore$ True for $n = 1$.

(B) Assume true for $n = k$, where $k \in \mathbb{N}$.
i.e. $5^k + 2 \times 11^k = 3M$ for $M \in \mathbb{N}$.

(C) Prove true for $n = k + 1$.
$$5^{k+1} + 2 \times 11^{k+1} = 5 \times 5^k + 2 \times 11^k \times 11$$
$$= 5(5^k + 2 \times 11^k) + 6 \times 2 \times 11^k$$

(D)
$$= 5(3M) + 3(6 \times 11^k) \quad \text{from } n = k \text{ assumption}$$
$$= 3(5 + 6 \times 11^k)$$
$$= 3N, \quad N \in \mathbb{N}$$
Hence, divisible by 3.
Shown true for $n = 1$, proven true for $n = k + 1$ by assuming true for $n = k$. Therefore, by Mathematical Induction, must be true for all positive integers n.

End of Section I

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Section II

90 marks

Attempt Questions 11-16

Allow about 2 hours and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a new writing booklet to answer this question.

(a) Given that $z = \sqrt{2}\cos\frac{\pi}{3} - \sqrt{2}\,i\sin\frac{\pi}{3}$, where z is a complex number.

(i) Express z in the form $x + iy$, where x and y are real numbers, in exact form. **2**

(ii) Evaluate z^{10}, leaving your answer in modulus argument form. **2**

(b) A line passes through the points $A(1, -1, 3)$ and $B(0, 5, 8)$.

(i) Show that the vector equation of the line is given by $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$ **2**

(ii) Find the midpoint of the segment joining AB. **1**

(iii) Hence, find the point P which divides the segment AB internally in the ratio 3:2. **2**

(c) Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \; dx$. **3**

(d) Prove using contradiction, that $\sqrt{4n-2}$ is irrational for all positive integers n. **3**

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Question 12 (15 marks) Use a new writing booklet to answer this question.

(a) Find the exact area bounded by the curve $y = \frac{x^2}{x^2+1}$, the lines $x = -1$, $x = \sqrt{3}$ and the x-axis. **3**

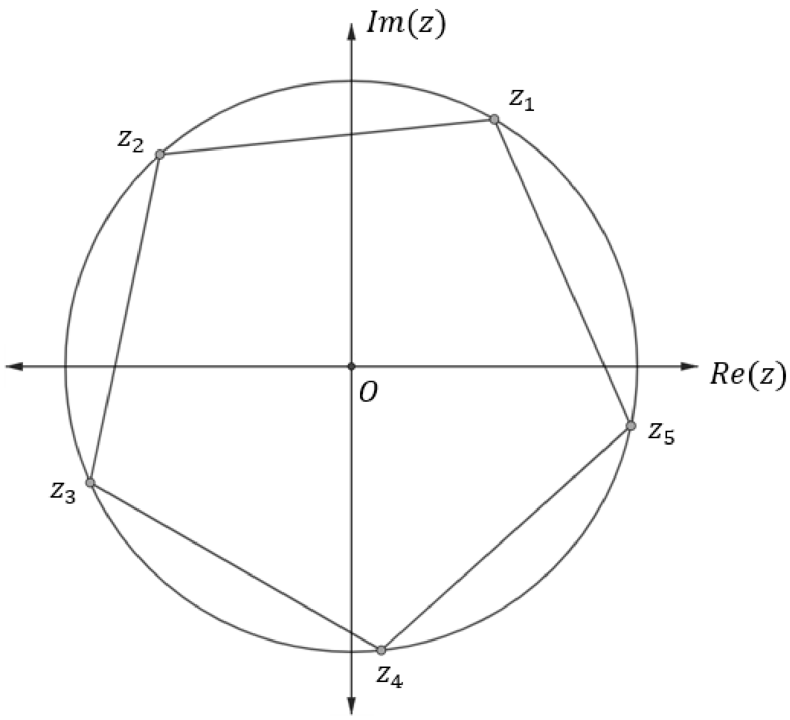
(b) The velocity v of a particle after time t, can be modelled using $v = \tan^{-1} t$, where v is in metres per second.

(i) Show using integration by parts, that $\int \tan^{-1} t \, dt = t \tan^{-1} t - \frac{1}{2}\log_e(1+t^2) + C$. **3**

(ii) Prove that $t \tan^{-1} t > \frac{1}{2}\log_e(1+t^2)$ for $t > 0$. **2**

(iii) Hence, if the particle starts at the origin, explain its motion as $t \to \infty$, and whether the particle will return to the origin. **1**

(c) A regular pentagon is inscribed inside the circumference of a circle with centre origin. Let z_1, z_2, z_3, z_4 and z_5 represent the vertices of the pentagon.



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(i) Given that $z_1 = 1 + i\sqrt{3}$, find z_2, z_3, z_4 and z_5 in exponential form. **4**

(ii) Find an equation in the form $z^5 = x + iy$, where z_1, z_2, z_3, z_4 and z_5 are the roots and x and y are real numbers. **2**

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Question 13 (15 marks) Use a new writing booklet to answer this question.

(a) The points $A(-3, 2, 6), B(-1, -3, -5), C(7, -1, -4)$ and $D(5, 4, 7)$ form the vertices of the quadrilateral $ABCD$.

(i) Deduce that $ABCD$ is a parallelogram by showing that $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$. **2**

(ii) Hence, find the point X where the diagonals of the parallelogram $ABCD$ intersect. **2**

(b) Let $z^4 = (\cos\theta + i\sin\theta)^4$ be a complex number.

(i) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ using De Moivre's Theorem and Binomial Expansion Theorem. **2**

(ii) Hence, by solving $16x^4 - 16x^2 + 1 = 0$ using the substitution $x = \cos\theta$, prove that $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{1}{16}$ **4**

(c) Find $\int \frac{16x-43}{x^3-4x^2-3x+18}\,dx$ **3**

(d) Let $\underset{\sim}{u}$ be the projection vector of the line segment $\underset{\sim}{r} = \lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$ where $0 \le \lambda \le 1$ onto the xy plane. Find the unit vector in the direction of $\underset{\sim}{u}$. **2**

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Question 14 (15 marks) Use a new writing booklet to answer this question.

(a) Consider the integral $I_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$ where n is an integer, $n \geq 0$.

(i) Show that $x^n = x^{n-2}(1+x^2) - x^{n-2}$. **1**

(ii) Hence, show that $I_n = \frac{1}{n-1} - I_{n-2}$ for $n \geq 2$. **2**

(iii) Evaluate I_{10} and show that $\pi < 3\frac{107}{315}$. **4**

(b) Let $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ be a complex number.

(i) Express z in the form $e^{i\theta}$ where θ is the argument of z. **1**

(ii) Evaluate z^i in exact form. **1**

(c) Solve $4z^3 - iz^2 - 4z + i = 0$ over the set of complex numbers. **3**

(d) Consider the following pair of statements P and Q.

P: $\underset{\sim}{a} = \lambda\,\underset{\sim}{b}$

Q: The vectors are parallel.

(i) Determine if the statements P and Q form an equivalence, that is $P \Leftrightarrow Q$. **1**

(ii) Write the contrapositive of $P \Rightarrow Q$ and explain why it is true. **2**

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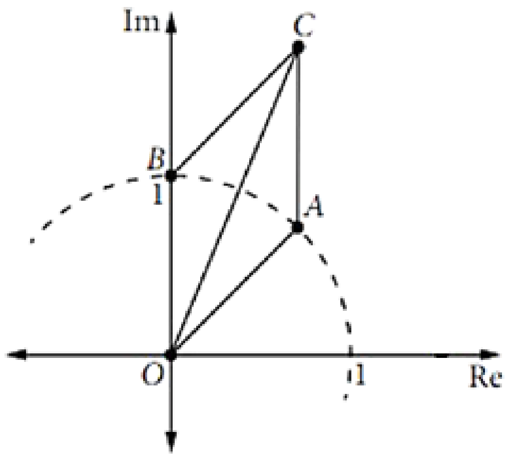
Question 15 (15 marks) Use a new writing booklet to answer this question.

(a) The AM-GM inequality for any two positive real numbers x and y is $\frac{x+y}{2} \geq \sqrt{xy}$. **3**

Use this result to prove that $(x+y)(y+z)(z+x) \geq 8xyz$ for x, y and z are positive real numbers.

(b) The points A and B in the complex plane correspond to the complex numbers $z = \frac{1}{\sqrt{2}}(1+i)$ and $w = i$, respectively. Let point C represent the complex number $z + w$.



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(i) Explain why the quadrilateral OACB forms a rhombus. **1**

(ii) Show that $\arg(z+w) = \frac{3\pi}{8}$ **2**

(iii) Hence, show that $\tan\frac{3\pi}{8} = \sqrt{2} + 1$ **2**

(c) Given that $y = e^x \sin x$.

(i) Show that $\sin\left(x + \frac{n\pi}{4}\right) + \cos(x + \frac{n\pi}{4}) = \sqrt{2}\sin(x + \frac{(n+1)\pi}{4})$ for n is an integer. **2**

(ii) Prove by Mathematical Induction that $\frac{d^n y}{dx^n} = 2^{\frac{n}{2}} e^x \sin(x + \frac{n\pi}{4})$ for all positive integers n. **4**

(iii) Hence, show that $2^4 \times \frac{d^k y}{dx^k} = \frac{d^{k+8} y}{dx^{k+8}}$, where k is a positive integer. **1**

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Question 16 (15 marks) Use a new writing booklet to answer this question.

(a) The fraction $\frac{22}{7}$ is a common rational approximation for π.

(i) Show that $x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$. **1**

(ii) Hence, prove that $\frac{22}{7}$ is an overestimate of π by evaluating the integral $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$. **3**

(b) Given that z and w are complex numbers where $w = \frac{z-2}{z-i}$, $w \neq 0$ is purely imaginary, sketch the locus of z **3**

(c) Consider the function $y = \ln x$ from $x = 1$ to $x = n$, where n is some positive integer.

(i) Use the Trapezoidal rule with $n-1$ sub-intervals to show that: **2**

$$\int_1^n \ln x \; dx > \frac{1}{2}\ln n + \ln(n-1)!$$

(ii) Hence, if $\int_1^n \ln x \; dx = n\ln n - n + 1$, show that: **2**

$$n! < n^{n+\frac{1}{2}}e^{1-n}$$

Question 16 continues on page 14

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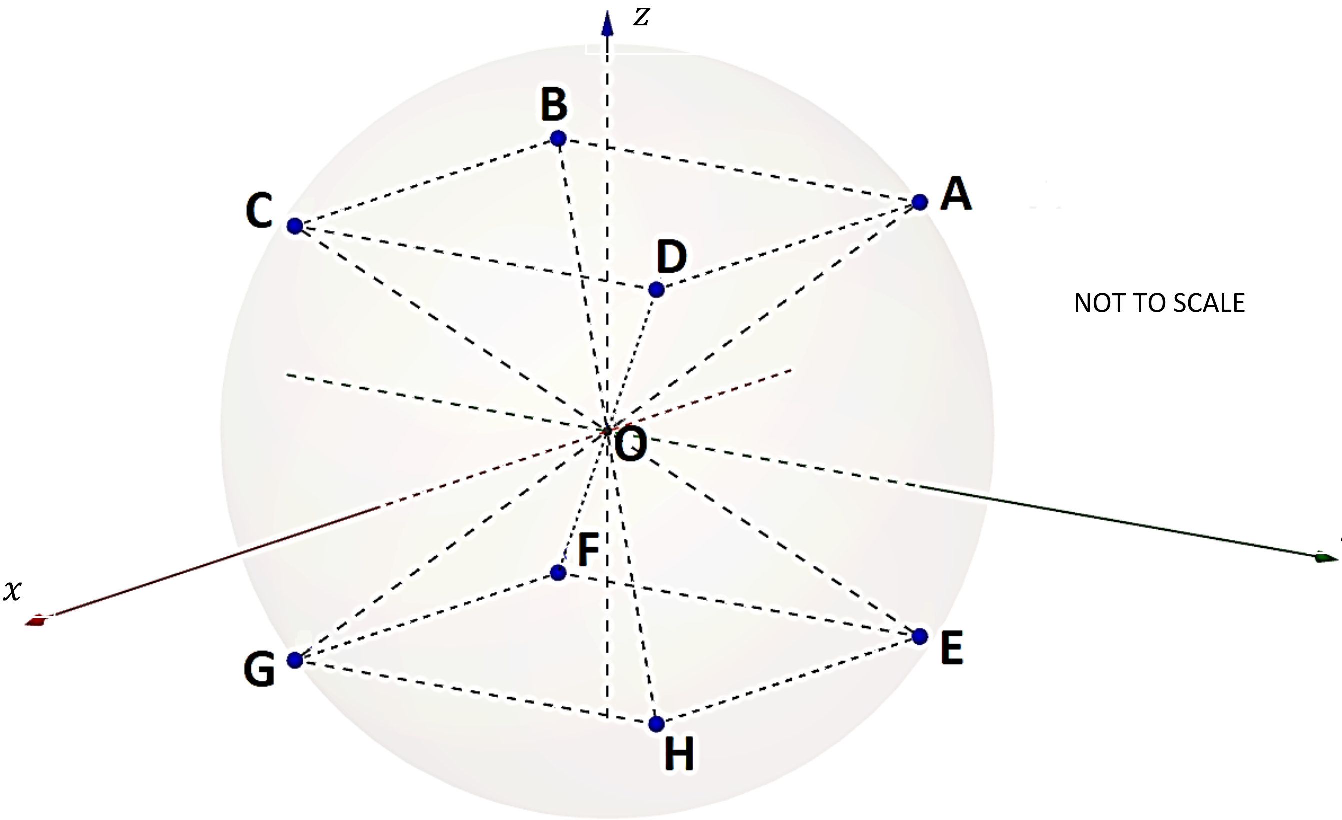
Question 16 (Continued)

(d) Two square pyramids OABCD and OEFGH are inscribed inside a sphere with centre $O(0,0,0)$ and radius 1 unit.
The points A, B, C, D, E, F, G and H all lie on the surface of the sphere and ABCDEFGH forms a cube.



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(i) Show that the longest diagonal in a cube with side length a units is $\sqrt{3}a$ units. **1**

(ii) Hence, derive that $a = \frac{2}{\sqrt{3}}$ for the cube ABCDEFGH. **1**

(iii) Hence, find the sum of the volume of the two square pyramids OABCD and OEFGH. **1**

(iv) Show that $\overrightarrow{OA} \cdot \overrightarrow{OC} = -\frac{1}{3}$. **1**

End of Section II

End of Assessment Task

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2023 **HIGHER SCHOOL CERTIFICATE EXAMINATION**

Mathematics Extension 2

Assessment Task 4 – Trial HSC

Section I - Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9
A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ✗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

correct

A ✗ B ✗ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

2023 Year 12 Extension 2 Mathematics Assessment Task 4
Marking Guidelines

Section I (1 mark each)

Multiple-choice Answer Key

Question	Answer
1	B
2	A
3	B
4	D
5	D
6	C
7	A
8	D
9	C
10	D

Question 1

$3i - \overline{z} = 3i - (3 + i)$

$= -3 + 2i$

Answer: B

Question 2

Centre $(1, 2, -2)$ and radius 3 implies $\left|\begin{pmatrix} x-1 \\ y-2 \\ z+2 \end{pmatrix}\right| = 3$

$(x-1)^2 + (y-2)^2 + (z+2)^2 = 9$

$x^2 + 2x + 1 + y^2 - 4y + 4 + z^2 + 4z + 4 = 9$

$x^2 + 2x + y^2 - 4y + z^2 + 4z = 0$

Answer: A

Question 3

Given $P \Rightarrow Q$, the converse is $Q \Rightarrow P$

So, "If y is even, then x is odd" is the converse statement.

Answer: B

Question 4

Perpendicular if $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$

$$\begin{pmatrix} 1.2 \\ 0 \\ k \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 0 \\ -10 \end{pmatrix} = 0$$

$1.2 \times 6 + 0 \times 0 + k \times -10 = 0$

$7.2 - 10k = 0$

$k = \dfrac{18}{25}$

Answer: D

Question 5

$\omega = \cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}$

$\omega^3 = \cos\dfrac{3\pi}{5} + i\sin\dfrac{3\pi}{5}$

$\omega^7 = \cos\dfrac{7\pi}{5} + i\sin\dfrac{7\pi}{5} = \overline{\omega^3}$

$\omega^9 = \cos\dfrac{9\pi}{5} + i\sin\dfrac{9\pi}{5} = \overline{\omega}$

$(z-\omega)(z-\overline{\omega})(z-\omega^3)(z-\overline{\omega^3})$

$(z^2 - z(\omega+\overline{\omega}) + \omega\overline{\omega})(z^2 - z\left(\omega^3 + \overline{\omega^3}\right) + \omega^3\overline{\omega^3})$

$(z^2 - z\big(2Re(\omega)\big) + |\omega|^2)(z^2 - z\big(2Re(\omega^3)\big) + |\omega^3|^2)$

$(z^2 - 2\cos\dfrac{\pi}{5}z + 1)(z^2 - 2\cos\dfrac{3\pi}{5}z + 1)$

$z^4 + z^3\left(-2\cos\dfrac{3\pi}{5} - 2\cos\dfrac{\pi}{5}\right) + z^2\left(2 + 4\cos\dfrac{\pi}{5}\cos\dfrac{3\pi}{5}\right) + z\left(-2\cos\dfrac{\pi}{5} - 2\cos\dfrac{3\pi}{5}\right) + 1$

$z^4 - z^3 + z^2 - z + 1$

Answer: D

Question 6

Let $u = x^n$ $v' = e^{2x}$

$u' = nx^{n-1}$ $v = \frac{1}{2}e^{2x}$

$$I_n = \int x^n e^{2x} dx = \frac{1}{2}x^n e^{2x} - \frac{1}{2}\int nx^{n-1}\, e^{2x} dx$$

$$= \frac{1}{2}x^n e^{2x} - \frac{1}{2}n\int x^{n-1}\, e^{2x} dx$$

$$= \frac{1}{2}x^n e^{2x} - \frac{1}{2}nI_{n-1}$$

Answer: C

Question 7

Let α and β be roots of the quadratic.

$\therefore \alpha + \beta = 4(1+i) = 4 + 4i$ and $\alpha\beta = 10i$

Hence, $\alpha = 3 + i$ and $\beta = 1 + 3i$ satisfies the sum and product of roots.

Answer: A

Question 8

Region inside circle centre at $z = i$ radius 1 unit including circumference is $|z - i| \leq 1$

Region between two angles excluding the boundaries, subtended from $z = -1 + i$ with positive and negative acute angle, is best represented with $-\frac{\pi}{6} < \arg\big(z - (-1 + i)\big) < \frac{\pi}{6}$

Hence, the intersection of the two regions mentioned above will yield the shaded region provided.

Answer: D

Question 9

Consider the parametric equations $x = \cos t$, $y = \sin t$ and $z = t$

For the xz projection, take $x = \cos t$ and $z = t$

Hence, $x = \cos z$ from $0 \leq z \leq 2\pi$ is the xz projection.

The only graph that best satisfies this is graph C.

Answer: C

Question 10

The algebraic error occurs in section D, when factorising out a common factor of 3 for the divisibility statement.

Answer: D

Section II

Question 11 (a)(i)

Criteria	Marks
• Correctly expresses in the form $x + iy$ with both real and imaginary parts shown correctly in exact form.	2
• Expresses in the form $x + iy$ with only the real or imaginary part shown correctly or not in exact form.	1

Sample answer:

$$z = \sqrt{2}\cos\frac{\pi}{3} - \sqrt{2}\, i \sin\frac{\pi}{3}$$

$$z = \sqrt{2}\left(\frac{1}{2}\right) - \sqrt{2}\left(\frac{\sqrt{3}}{2}\right) i$$

$$z = \frac{\sqrt{2}}{2} - \frac{\sqrt{6}}{2} i$$

Question 11 (a)(ii)

Criteria	Marks
• Correctly evaluates z^{10} by first turning z into modulus argument form, then applying De Moivre's Theorem.	2
• Modifies z into modulus argument form but does not correctly apply De Moivre's theorem to evaluate z^{10} or Applies De Moivre's Theorem to evaluate z^{10} but does not modify z into modulus argument form first.	1

Sample answer:

$$z = \sqrt{2}(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right))$$

$$z^{10} = (\sqrt{2})^{10}(\cos\left(-\frac{10\pi}{3}\right) + i \sin\left(-\frac{10\pi}{3}\right))$$

$$z^{10} = 2^5(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right))$$

Question 11 (b)(i)

Criteria	Marks
• Correctly constructs and proves the vector equation of the line to be $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$ by finding vector $\underset{\sim}{a}$ and combining with the direction vector.	2
• Correctly proves and shows the direction vector to be $\begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$.	1

Sample answer:

Direction vector $\underset{\sim}{b} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$

Vector $\underset{\sim}{a}$ on the line can be found using $\overrightarrow{OA}$

Hence, $\underset{\sim}{a} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

Therefore, vector equation of the line is $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix}$.

Question 11 (b)(ii)

Criteria	Marks
• Correctly calculates the midpoint of segment AB by letting $\lambda = 0.5$	1

Sample answer:

Let $\lambda = 0.5$

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + 0.5 \begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 2 \\ 5.5 \end{pmatrix}$$

$M = (0.5, 2, 5.5)$

Question 11 (b)(iii)

Criteria	Marks
• Substitutes the correct value of $\lambda = \frac{m}{m+n} = \frac{3}{5}$ into the vector equation of a line to product the point P which divides the segment in the ratio 3:2 internally.	2
• Attempts to substitute a value of λ between 0 and 1 into vector equation of a line to product a point P.	1

Sample answer:

Ratio is 3:2 internally, hence $m = 3$ and $n = 2$

Let $\lambda = \frac{m}{m+n} = \frac{3}{5}$

$$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \frac{3}{5}\begin{pmatrix} -1 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0.4 \\ 2.6 \\ 6 \end{pmatrix}$$

Point $P(0.4, 2.6, 6)$

Question 11 (c)

Criteria	Marks
• Correctly evaluates the definite integral.	3
• Uses a u-substitution or appropriate to simplify the integral to make u the subject.	2
• Identifies the correct trigonometric substitution $u = \tan x$.	1

Sample answer:

Let $u = \tan x$

$$\frac{du}{dx} = \sec^2 x$$

$$du = \sec^2 x \, dx$$

$$x = 0, u = 0$$

$$x = \frac{\pi}{4}, u = 1$$

$$\int_0^1 u^3 \, du = \left[\frac{u^4}{4}\right]_0^1$$

$$= \frac{1}{4}$$

Question 11 (d)

Criteria	Marks
• Correctly demonstrates the contradiction in the initial premise with full working out shown and concludes using proof by contradiction that $\sqrt{4n-2}$ is irrational.	3
• Identifies and uses the factors of p^2 and q^2 and the idea of even/odd integers, to establish the base premise leading up to the contradiction statement.	2
• Establishes the correct initial premise for a contradiction based proof.	1

Sample answer:

Assume that $\sqrt{4n-2}$ is rational, that is $\sqrt{4n-2} = \frac{p}{q}$ where p and q have highest common factor 1 and $q \neq 0$.

$$\therefore 4n - 2 = \frac{p^2}{q^2}$$

$$4n - 2 = \frac{p^2}{q^2}$$

Since LHS is an even integer, then $\frac{p^2}{q^2}$ must also be an even integer. Since highest common factor of p and q is 1, then highest common factor of p^2 and q^2 is also 1. Therefore, $q^2 = 1$ such that $\frac{p^2}{q^2}$ is an even integer.

$$\therefore 4n - 2 = p^2$$

If p is odd, then p^2 is odd. But $4n - 2$ is even, hence, p must be even.

If p is even, then p^2 is even, i.e. $p = 2k$ where $k \in \mathbb{N}$.

$$\therefore 4n - 2 = (2k)^2$$

$$4n - 2 = 4k^2$$

$$n - \frac{1}{2} = k^2$$

$$k = \sqrt{n - \frac{1}{2}}$$

$\therefore k$ is not an integer.

But, if p is even, k needs to be an integer, so there is a contradiction for all cases of p.

Hence, our initial premise of $\sqrt{4n-2}$ being rational is false. Therefore, using proof by contradiction, $\sqrt{4n-2}$ must be irrational for all positive integers of n.

Question 12 (a)

Criteria	Marks
• Correctly evaluates the definite integral to calculate the bounded area.	3
• Separates the rational function into a simpler expression $1-\frac{1}{x^2+1}$.	2
• Establishes the correct integral for calculating the bounded area.	1

Sample answer:

Area $=\int_{-1}^{\sqrt{3}} \frac{x^2}{x^2+1}\,dx$

$$=\int_{-1}^{\sqrt{3}} 1-\frac{1}{x^2+1}\,dx$$

$$=[x-\tan^{-1}x]_{-1}^{\sqrt{3}}$$

$$=\left(\sqrt{3}-\frac{\pi}{3}\right)-\left(-1+\frac{\pi}{4}\right)$$

$=\left(\sqrt{3}+1-\frac{7\pi}{12}\right)$ units^2

Question 12 (b)(i)

Criteria	Marks
• Correctly simplifies the resultant integral to demonstrate the required proof.	3
• Constructs a correct resultant integral using one application of integration by parts.	2
• Correctly identifies the elements required to initiate integration by parts.	1

Sample answer:

Let $u=\tan^{-1}t$ and $w'=1$

$u'=\frac{1}{1+t^2}$ and $w=t$

$$\int \tan^{-1}t\ dt = t\tan^{-1}t-\int \frac{t}{1+t^2}\ dt$$

$$=t\tan^{-1}t-\frac{1}{2}\int\frac{2t}{1+t^2}\ dt$$

$$=t\tan^{-1}t-\frac{1}{2}\log_e(1+t^2)+C$$

Question 12 (b)(ii)

Criteria	Marks
• Correctly proves the statement by considering the derivative of $t\tan^{-1}t - \frac{1}{2}\log_e(1+t^2)$.	2
• Establishes the required to prove statement as $t\tan^{-1}t - \frac{1}{2}\log_e(1+t^2) > 0$.	1

Sample answer:

$t\tan^{-1}t > \frac{1}{2}\log_e(1+t^2) \text{ for } t > 0 \text{ if and only if } t\tan^{-1}t - \frac{1}{2}\log_e(1+t^2) > 0 \text{ for } t > 0$

$\text{At } t = 0,\ t\tan^{-1}t - \frac{1}{2}\log_e(1+t^2) = 0$

However, $\frac{d}{dt}\left(t\tan^{-1}t - \frac{1}{2}\log_e(1+t^2)\right) = \tan^{-1}t > 0$ for $t > 0$, hence the curve is monotonic increasing starting from 0.

$\therefore\ t\tan^{-1}t - \frac{1}{2}\log_e(1+t^2) > 0$ for $t > 0$ which implies that $t\tan^{-1}t > \frac{1}{2}\log_e(1+t^2)$ for $t > 0$.

Proof complete.

Question 12 (b)(iii)

Criteria	Marks
• Correctly explains the motion of the particle and whether it will return to the origin.	1

Sample answer:

$v \to \frac{\pi}{2}$ as $t \to \infty$ and $x \to \infty$ as $t \to \infty$.

The particle will move further away from the origin at a velocity that will remain constant. The particle will also never return to the origin since displacement is monotonic increasing.

Question 12 (c)(i)

Criteria	Marks
• Correctly finds all of z_2, z_3, z_4 and z_5 in exponential form.	4
• Correctly finds one of the required complex numbers z_2, z_3, z_4 and z_5 in exponential form.	3
• Recognises roots are equally spaced apart on the circle with difference in argument of $\frac{2\pi}{5}$ radians.	2
• Identifies the argument and modulus of z_1	1

Sample answer:

$$|z_1| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\arg z_1 = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

$$z_1 = 2e^{i\frac{\pi}{3}}$$

Complex numbers are all equally spaced apart by $\frac{2\pi}{n} = \frac{2\pi}{5}$ radians.

$$\therefore z_2 = 2e^{i(\frac{\pi}{3}+\frac{2\pi}{5})} = 2e^{i\frac{11\pi}{15}}$$

$$z_3 = 2e^{i\left(\frac{\pi}{3}-\frac{6\pi}{5}\right)} = 2e^{i\left(\frac{-13\pi}{15}\right)}$$

$$z_4 = 2e^{i(\frac{\pi}{3}-\frac{4\pi}{5})} = 2e^{i\left(\frac{-7\pi}{15}\right)}$$

$$z_5 = 2e^{i(\frac{\pi}{3}-\frac{2\pi}{5})} = 2e^{i\left(\frac{-\pi}{15}\right)}$$

Question 12 (c)(ii)

Criteria	Marks
• Correctly finds the equation z^5 in $x + iy$ form where x and y are real numbers.	2
• Recognises z_1, z_2, z_3, z_4 and z_5 as the roots to some equation $z^5 = a + ib$ and applies De Moivre's Theorem on one of the roots to get an expression for z^5.	1

Sample answer:

Let $z_1 = 2e^{i\frac{\pi}{3}}$ be a root to z^5

$$\therefore z^5 = (2e^{i\frac{\pi}{3}})^5$$

$$= 2^5 e^{i\frac{5\pi}{3}}$$

$$= 32(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3})$$

$$z^5 = 16 - 16\sqrt{3}i$$

Question 13 (a)(i)

Criteria	Marks
• Correctly demonstrates that both $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{AD} = \overrightarrow{BC}$, then makes an appropriate conclusion as to why this is a parallelogram.	2
• Demonstrates that $\overrightarrow{AB} = \overrightarrow{DC}$ OR $\overrightarrow{AD} = \overrightarrow{BC}$ only, using vector calculations.	1

Sample answer:

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -11 \end{pmatrix}$$

$$\overrightarrow{DC} = \overrightarrow{OC} - \overrightarrow{OD} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -11 \end{pmatrix}$$

Hence, $\overrightarrow{AB} = \overrightarrow{DC}$

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} -3 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 1 \end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 7 \\ -1 \\ -4 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 8 \\ 2 \\ 1 \end{pmatrix}$$

Hence, $\overrightarrow{AD} = \overrightarrow{BC}$

Therefore, opposite sides in the quadrilateral are parallel since they are the same vectors. The quadrilateral is a parallelogram.

Question 13 (a)(ii)

Criteria	Marks
• Correctly calculates the point X using the idea that diagonals in a parallelogram bisect each other.	2
• Calculates the vector representing one of the diagonals of the quadrilateral.	1

Sample answer:

$$\overrightarrow{BD} = \overrightarrow{OD} - \overrightarrow{OB} = \begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix} - \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 12 \end{pmatrix}$$

$$\overrightarrow{OX} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BD} = \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 6 \\ 7 \\ 12 \end{pmatrix} = \begin{pmatrix} 2 \\ 0.5 \\ 1 \end{pmatrix}$$

Point $X(2, 0.5, 1)$

Question 13 (b)(i)

Criteria	Marks
• Compares the real coefficients in both expressions to yield the desired trigonometric identity.	2
• Expresses z^4 using De Moivre's Theorem and Binomial Expansion Theorem.	1

Sample answer:

$z^4 = \cos 4\theta + i\sin 4\theta$ (Using De Moivre's Theorem)

$z^4 = \cos^4\theta + 4\cos^3\theta\, i\sin\theta + 6\cos^2\theta\, i^2\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + i^4\sin^4\theta$ (Using Binomial Expansion Theorem)

Equating the real parts of both expansions yields:

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$

$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$

$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1$$

Proof complete

Question 13 (b)(ii)

Criteria	Marks
• Applies product of roots of a polynomial with degree 4 to yield the desired trigonometric identity.	4
• Solves the trigonometric equation and yields four unique solutions for θ.	3
• Uses the identity from part i) to simplify the trigonometric equations	2
• Substitutes $x = \cos\theta$ to yield a trigonometric equation.	1

Sample answer:

Let $x = \cos\theta$

$$16\cos^4\theta - 16\cos^2\theta + 1 = 0$$

$$2(8\cos^4\theta - 8\cos^2\theta + 1) - 1 = 0$$

$2\cos 4\theta - 1 = 0$ using the result from part i)

$$\cos 4\theta = \frac{1}{2}$$

$$4\theta = \cdots, \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}, \ldots$$

$$\theta = \cdots, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \ldots$$

These four values are unique, solutions repeat as we move into the next period/cycle of solutions.

$\therefore$ Roots are $x_1 = \cos\frac{\pi}{12}, x_2 = \cos\frac{5\pi}{12}, x_3 = \cos\frac{7\pi}{12}, x_4 = \cos\frac{11\pi}{12}$

The product of roots $x_1x_2x_3x_4 = \frac{e}{a} = \frac{1}{16}$ from the polynomial $16x^4 - 16x^2 + 1 = 0$.

Hence, $\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{1}{16}$.

Proof complete.

Question 13 (c)

Criteria	Marks
• Correctly finds the indefinite integral by integrating the partial fraction result.	3
• Decomposes the rational function into partial fractions.	2
• Factorises the denominator of the rational function	1

Sample answer:

$$x^3 - 4x^2 - 3x + 18 = (x-3)(x^2 - x - 6)$$

$$= (x-3)(x-3)(x+2)$$

$$= (x-3)^2(x+2)$$

$$\frac{16x - 43}{(x-3)^2(x+2)} = \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x+2)}$$

$$16x - 43 = A(x-3)(x+2) + B(x+2) + C(x-3)^2$$

Let $x = 3, 5 = B(3+2), \therefore B = 1$

Let $x = -2, -75 = C(-2-3)^2, \therefore C = -3$

Let $x = 0, -43 = A(-3)(2) + 1(2) - 3(-3)^2, \therefore A = 3$

$$\frac{16x - 43}{(x-3)^2(x+2)} = \frac{3}{(x-3)} + \frac{1}{(x-3)^2} - \frac{3}{(x+2)}$$

$$\int \frac{16x - 43}{x^3 - 4x^2 - 3x + 18}dx = \int \frac{3}{(x-3)} + \frac{1}{(x-3)^2} - \frac{3}{(x+2)}dx$$

$$= 3\ln|x-3| - \frac{1}{(x-3)} - 3\ln|x+2| + C$$

Question 13 (d)

Criteria	Marks
• Correctly finds the unit vector in the direction of $\underset{\sim}{u}$.	2
• Finds the vector projection $\underset{\sim}{u}$ on the xy plane.	1

Sample answer:

The line segment $\underset{\sim}{r} = \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ where $0 \leq \lambda \leq 1$, starts at the origin and ends at the point $(2,3,1)$. This means the vector representing the projection on the xy plane is $\underset{\sim}{u} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

Hence, $\underset{\sim}{\hat{u}} = \frac{1}{\sqrt{13}} \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$.

Question 14 (a)(i)

Criteria	Marks
• Correctly demonstrates that LHS = RHS in the provided algebraic statement.	1

Sample answer:

$$RHS = x^{n-2}(1 + x^2) - x^{n-2}$$

$$= x^{n-2} + x^n - x^{n-2}$$

$$= x^n$$

$$= LHS$$

Hence, $LHS = RHS$. Proof complete.

Question 14 (a)(ii)

Criteria	Marks
• Correctly shows the recursive formula with full working out shown.	2
• Applies the result from part i) to simplify the integral into $\int_0^1 x^{n-2} - \frac{x^{n-2}}{1+x^2}\, dx$	1

Sample answer:

$$I_n = \int_0^1 \frac{x^n}{1+x^2} dx$$

$$= \int_0^1 \frac{x^{n-2}(1+x^2) - x^{n-2}}{1+x^2}\, dx$$

$$= \int_0^1 x^{n-2} - \frac{x^{n-2}}{1+x^2}\, dx$$

$$= \int_0^1 x^{n-2}\, dx - \int_0^1 \frac{x^{n-2}}{1+x^2}\, dx$$

$$= \left[\frac{x^{n-1}}{n-1}\right]_0^1 - I_{n-2}$$

$= \frac{1}{n-1} - I_{n-2}$, for $n \geq 2$.

Proof complete

Question 14 (a)(iii)

Criteria	Marks
• Correctly states that $I_{10} > 0$ and rearranges to yield the desired result.	4
• Evaluates I_{10} correctly.	3
• Evaluates an intermediate term leading up to I_{10}	2
• Evaluates I_0 correctly.	1

Sample answer:

$$I_0 = \int_0^1 \frac{x^0}{1+x^2}\, dx = \int_0^1 \frac{1}{1+x^2}\, dx = [\tan^{-1} x]_0^1 = \frac{\pi}{4}$$

$$I_2 = \frac{1}{2-1} - I_0 = 1 - \frac{\pi}{4}$$

$$I_4 = \frac{1}{4-1} - I_2 = \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) = -\frac{2}{3} + \frac{\pi}{4}$$

$$I_6 = \frac{1}{6-1} - I_4 = \frac{1}{5} - \left(-\frac{2}{3} + \frac{\pi}{4}\right) = \frac{13}{15} - \frac{\pi}{4}$$

$$I_8 = \frac{1}{8-1} - I_6 = \frac{1}{7} - \left(\frac{13}{15} - \frac{\pi}{4}\right) = -\frac{76}{105} + \frac{\pi}{4}$$

$$I_{10} = \frac{1}{10-1} - I_8 = \frac{1}{9} - \left(-\frac{76}{105} + \frac{\pi}{4}\right) = \frac{263}{315} - \frac{\pi}{4}$$

Since $\frac{x^n}{1+x^2} > 0$ for n is even, then $I_{10} > 0$.

Hence, $\frac{263}{315} - \frac{\pi}{4} > 0$

$$\frac{\pi}{4} < \frac{263}{315}$$

$$\pi < \frac{1052}{315}$$

$$\pi < 3\frac{107}{315}$$

Proof complete

Question 14 (b)(i)

Criteria	Marks
• Correctly expresses z in the form $e^{i\theta}$.	1

Sample answer:

$$\arg z = \tan^{-1}\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)} = \frac{\pi}{3}$$

$$|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$$

$$z = e^{i\frac{\pi}{3}}$$

Question 14 (b)(ii)

Criteria	Marks
• Correctly evaluates z^i in exact form.	1

Sample answer:

$$z^i = (e^{i\frac{\pi}{3}})^i$$

$$= e^{i^2\frac{\pi}{3}}$$

$$= e^{-\frac{\pi}{3}}$$

Question 14 (c)

Criteria	Marks
• Correctly finds all zeros of the polynomial over the set of complex numbers.	3
• Factorises and/or finds another zero using remainder theorem.	2
• Uses remainder theorem to identify one zero of the polynomial.	1

Sample answer:

Multiply each term by i.

$4iz^3 + z^2 - 4iz - 1 = 0$

Test $z = 1$ using remainder theorem.

$4i(1)^3 + (1)^2 - 4i(1) - 1 = 4i + 1 - 4i - 1 = 0$

So, $z = 1$ is a root of the polynomial.

Test $z = -1$ using remainder theorem.

$4i(-1)^3 + (-1)^2 - 4i(-1) - 1 = -4i + 1 + 4i - 1 = 0$

So, $z = -1$ is a root of the polynomial.

Test $z = \frac{1}{4i}$ using remainder theorem.

$$4i\left(\frac{1}{4i}\right)^3 + \left(\frac{1}{4i}\right)^2 - 4i\left(\frac{1}{4i}\right) - 1 = -\frac{1}{16} - \frac{1}{16} - 1 - 1 = -\frac{17}{8} \neq 0$$

∴ Not a zero.

Test $z = -\frac{1}{4i}$ using remainder theorem.

$$4i\left(-\frac{1}{4i}\right)^3 + \left(-\frac{1}{4i}\right)^2 - 4i\left(-\frac{1}{4i}\right) - 1 = \frac{1}{16} - \frac{1}{16} + 1 - 1 = 0$$

So, $z = -\frac{1}{4i}$ is a root of the polynomial.

Since polynomial is of degree 3, all roots have been found.

Hence, solutions to $4z^3 - iz^2 - 4z + i = 0$ are:

$z = 1, -1$ and $\frac{i}{4}$.

Question 14 (d)(i)

Criteria	Marks
• Provides correct reasoning for why the statements form an equivalence.	1

Sample answer:

$P \Rightarrow Q$ is true since if vectors are scalar multiples of each other, then they must be parallel.

Similarly, $Q \Rightarrow P$ is true since if vectors are parallel, then they must be scalar multiples of each other.

Hence, $P \Leftrightarrow Q$, they form an equivalence.

Question 14 (d)(ii)

Criteria	Marks
• Provides correct reasoning for why the contrapositive is true.	2
• Writes the contrapositive of $P \Rightarrow Q$ correctly.	1

Sample answer:

The contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$ which says "If vectors are not parallel, then they are not scalar multiples of each other"

This statement is also true, since the condition for two vectors to be parallel must involve them being scalar multiples of each other.

Question 15 (a)

Criteria	Marks
• Multiplies the three inequalities together to correctly prove the required inequality result.	3
• Constructs three correct inequalities using the AM-GM inequality for $(x + y)$, $(x + z)$ and $(y + z)$.	2
• Uses the AM-GM inequality to write an inequality statement for either $(x + y), (x + z)$ or $(y + z)$.	1

Sample answer:

$$\frac{x + y}{2} \geq \sqrt{xy}$$

$$\frac{y + z}{2} \geq \sqrt{yz}$$

$$\frac{z + x}{2} \geq \sqrt{zx}$$

Multiplying the three inequalities together, preserves the inequality, since both LHS and RHS are positive numbers in each inequality.

$$\therefore \left(\frac{x+y}{2}\right)\left(\frac{y+z}{2}\right)\left(\frac{z+x}{2}\right) \geq \sqrt{xy}\sqrt{yz}\sqrt{zx}$$

$$\frac{(x+y)(y+z)(z+x)}{8} \geq \sqrt{x^2y^2z^2}$$

$$\frac{(x+y)(y+z)(z+x)}{8} \geq xyz$$

$$(x+y)(y+z)(z+x) \geq 8xyz$$

Proof complete.

Question 15 (b)(i)

Criteria	Marks
• Provides correct reasoning explaining why OACB is a rhombus	1

Sample answer:

Side OA = Side OB since equal radii on unit circle as shown in provided diagram.

Side BC = Side OA and Side AC = Side OB, since OC is OA + OB.

Hence, OACB is a rhombus as all sides are equal in length and opposite sides are parallel.

Question 15 (b)(ii)

Criteria	Marks
• Correctly shows the argument is $\frac{3\pi}{8}$, using the property of diagonals bisect angles in a rhombus.	2
• Evaluates the argument of z correctly, or finds the size of $\angle BOA$.	1

Sample answer:

$\arg(z+w) = \arg z + \frac{1}{2}\angle BOA$

$\arg z = \frac{\pi}{4}, \therefore \angle BOA = \frac{\pi}{4}$.

Hence, $\arg(z+w) = \frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{3\pi}{8}$ (Diagonals bisect angles in a rhombus)

Proof complete.

Question 15 (b)(iii)

Criteria	Marks
• Correctly proves that $\tan\frac{3\pi}{8} = \sqrt{2} + 1$ using the established identity.	2
• Recognise and establishes that $\tan^{-1}\left(\frac{y}{x}\right) = \arg\,(x + iy)$ and links this to the result of $\arg(z + w)$, including evaluating $z + w$ in $x + iy$ form.	1

Sample answer:

Recall that $\arg(x + iy) = \tan^{-1}\frac{y}{x}$ for acute angles.

$$\tan(\arg(x + iy)) = \frac{y}{x}$$

Now $z + w = \frac{1}{\sqrt{2}}(1 + i) + i = \frac{1}{\sqrt{2}} + \left(1 + \frac{1}{\sqrt{2}}\right)i$

$$\tan(\arg(z + w)) = \frac{\left(1 + \frac{1}{\sqrt{2}}\right)}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$\tan\left(\frac{3\pi}{8}\right) = \frac{\sqrt{2} + 1}{\sqrt{2}} \times \sqrt{2}$$

$$\tan\left(\frac{3\pi}{8}\right) = \sqrt{2} + 1$$

Proof complete.

Question 15 (c)(i)

Criteria	Marks
• Correctly constructs a LHS = RHS style proof with full working out shown to prove the trigonometric identity.	2
• Applies a correct trigonometric identity to the RHS expression when initiating a LHS = RHS style proof.	1

Sample answer:

$$RHS = \sqrt{2}\sin\left(x + \frac{(n + 1)\pi}{4}\right)$$

$$= \sqrt{2}(\sin x\cos\left(\frac{(n + 1)\pi}{4}\right) + \cos x\sin\left(\frac{(n + 1)\pi}{4}\right))$$

$$= \sqrt{2}(\sin x\cos\left(\frac{n\pi}{4} + \frac{\pi}{4}\right) + \cos x\sin(\frac{n\pi}{4} + \frac{\pi}{4}))$$

$$= \sqrt{2}\left(\sin x\left(\frac{1}{\sqrt{2}}\cos\frac{n\pi}{4} - \frac{1}{\sqrt{2}}\sin\frac{n\pi}{4}\right) + \cos x\left(\frac{1}{\sqrt{2}}\sin\frac{n\pi}{4} + \frac{1}{\sqrt{2}}\cos\frac{n\pi}{4}\right)\right)$$

$= \sin x \cos\frac{n\pi}{4} - \sin x \sin\frac{n\pi}{4} + \cos x \sin\frac{n\pi}{4} + \cos x \cos\frac{n\pi}{4}$

$= \left(\sin x \cos\frac{n\pi}{4} + \cos x \sin\frac{n\pi}{4}\right) + (\cos x \cos\frac{n\pi}{4} - \sin x \sin\frac{n\pi}{4})$

$= \sin\left(x + \frac{n\pi}{4}\right) + \cos(x + \frac{n\pi}{4})$

$= LHS$

$\therefore LHS = RHS$

Proof complete

Question 15 (c)(ii)

Criteria	Marks
• Correctly proves the $n = k + 1$ case and makes an appropriate conclusion for mathematical induction.	4
• Correctly applies the $n = k$ assumption as part of the proof for the $n = k + 1$ case.	3
• Correctly writes the $n = k$ assumption and the statement to prove for $n = k + 1$ case.	2
• Correctly shows true for the $n = 1$ case.	1

Sample answer:

Show true for $n = 1$

$LHS = \frac{dy}{dx}$

$= e^x \sin x + e^x \cos x$

$RHS = \sqrt{2}e^x \sin\left(x + \frac{\pi}{4}\right)$

$= e^x(\sqrt{2}\sin(x + \frac{\pi}{4}))$

$= e^x(\sin x + \cos x)$ using identity from part i)

$= e^x \sin x + e^x \cos x$

Hence, $LHS = RHS$

True for $n = 1$

Assume true for $n = k$, where $k \in \mathbb{N}$

i.e. $\frac{d^k y}{dx^k} = 2^{\frac{k}{2}} e^x \sin(x + \frac{k\pi}{4})$

Prove true for $n = k + 1$

RTP: $\frac{d^{k+1}y}{dx^{k+1}} = 2^{\frac{k+1}{2}} e^x \sin(x + \frac{(k+1)\pi}{4})$

$$LHS = \frac{d^{k+1}y}{dx^{k+1}}$$

$$= \frac{d}{dx}(\frac{d^k y}{dx^k})$$

$= \frac{d}{dx}(2^{\frac{k}{2}} e^x \sin\left(x + \frac{k\pi}{4}\right))$ using $n = k$ assumption.

$$= 2^{\frac{k}{2}}(e^x \sin\left(x + \frac{k\pi}{4}\right) + e^x \cos\left(x + \frac{k\pi}{4}\right))$$

$$= 2^{\frac{k}{2}} e^x (\sin\left(x + \frac{k\pi}{4}\right) + \cos\left(x + \frac{k\pi}{4}\right))$$

$= 2^{\frac{k}{2}} e^x (\sqrt{2} \sin\left(x + \frac{(k+1)\pi}{4}\right))$ using identity from part i)

$$= 2^{\frac{k}{2}} \times 2^{\frac{1}{2}} \times e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$$

$$= 2^{\frac{k+1}{2}} e^x \sin\left(x + \frac{(k+1)\pi}{4}\right)$$

$$= RHS$$

True for $n = k + 1$

Hence, shown true for $n = 1$, proven true for $n = k + 1$ by assuming true for $n = k$. Therefore, by mathematical induction, must be true for all positive integers n.

Question 15 (c)(iii)

Criteria	Marks
• Correctly uses the induction identity to prove the given statement.	1

Sample answer:

$$LHS = 2^4 \times 2^{\frac{k}{2}} e^x \sin\left(x + \frac{k\pi}{4}\right)$$

$$= 2^{\frac{k+8}{2}} e^x \sin\left(x + \frac{k\pi}{4}\right)$$

$$= 2^{\frac{k+8}{2}} e^x \sin\left(x + \frac{k\pi}{4} + 2\pi\right)$$

$$= 2^{\frac{k+8}{2}} e^x \sin\left(x + \frac{k\pi + 8\pi}{4}\right)$$

$$= 2^{\frac{k+8}{2}} e^x \sin\left(x + \frac{(k+8)\pi}{4}\right)$$

$$= \frac{d^{k+8}y}{dx^{k+8}}$$

$$= RHS$$

Proof complete.

Question 16 (a)(i)

Criteria	Marks
• Correctly uses binomial expansion or equivalent to show that $x^4(1-x)^4 = x^8 - 4x^7 + 6x^6 - 4x^5 + x^4$.	1

Sample answer:

From binomial expansion, $(1-x)^4 = 1 - 4x + 6x^2 - 4x^3 + x^4$

$$\begin{aligned} LHS &= x^4(1-x)^4 \\ &= x^4(1 - 4x + 6x^2 - 4x^3 + x^4) \\ &= x^8 - 4x^7 + 6x^6 - 4x^5 + x^4 \\ &= RHS \end{aligned}$$

Proof complete

Question 16 (a)(ii)

Criteria	Marks
• Recognises that $\frac{x^4(1-x)^4}{1+x^2} > 0$ always, due to even powers and uses this identity to correctly prove that $\frac{22}{7} > \pi$.	3
• Evaluates the definite integral correctly to yield $\frac{22}{7} - \pi$.	2
• Attempts long division and correctly expresses the integral as $\int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\,dx$	1

Sample answer:

$$
\begin{array}{r l}
 & x^6 - 4x^5 + 5x^4 - 4x^2 + 4 \\
1 + x^2 & \overline{\big) x^8 - 4x^7 + 6x^6 - 4x^5 + x^4} \\
 & x^8 \qquad\quad + x^6 \\
\hline
 & -4x^7 + 5x^6 - 4x^5 + x^4 \\
 & -4x^7 \qquad\quad - 4x^5 \\
\hline
 & 5x^6 \qquad + x^4 \\
 & 5x^6 \qquad + 5x^4 \\
\hline
 & -4x^4 \\
 & -4x^4 - 4x^2 \\
\hline
 & 4x^2 \\
 & 4x^2 + 4 \\
\hline
 & -4
\end{array}
$$

$$\therefore \frac{x^4(1-x)^4}{1+x^2} = x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}$$

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \int_0^1 x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2}\,dx$$

$$= \left[\frac{x^7}{7} - \frac{2x^6}{3} + x^5 - \frac{4x^3}{3} + 4x - 4\tan^{-1}x\right]_0^1$$

$$= \left(\frac{22}{7} - \pi\right) - (0)$$

$$= \frac{22}{7} - \pi$$

Since $\frac{x^4(1-x)^4}{1+x^2} > 0$ for $0 < x < 1$, then $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\ dx > 0$.

Hence, $\frac{22}{7} - \pi > 0$

$$\frac{22}{7} > \pi$$

Therefore, $\frac{22}{7}$ is an overestimate of π.

Proof complete

Question 16 (b)

Criteria	Marks
• Correctly sketches the locus of z, which is a circle with both restrictions shown.	3
• Uses the property that w is purely imaginary to yield the correct cartesian equation, which is a circle.	2
• Uses $z = x + iy$ and attempts to express w in $x + iy$ form by realising the denominator.	1

Sample answer:

Let $z = x + iy$ where x and y are real numbers.

$$w = \frac{(x-2) + iy}{x + i(y-1)}$$

$$w = \frac{(x-2) + iy}{x + i(y-1)} \times \frac{x - i(y-1)}{x - i(y-1)}$$

$$w = \frac{\big((x-2) + iy\big)\big(x - i(y-1)\big)}{x^2 + (y-1)^2}$$

$$w = \frac{x(x-2) + y(y-1) + i(xy - (x-2)(y-1))}{x^2 + (y-1)^2}$$

Since w is purely imaginary, then $Re(w) = 0$.

$$Re(w) = \frac{x(x-2) + y(y-1)}{x^2 + (y-1)^2} = 0$$

$$x(x-2) + y(y-1) = 0$$

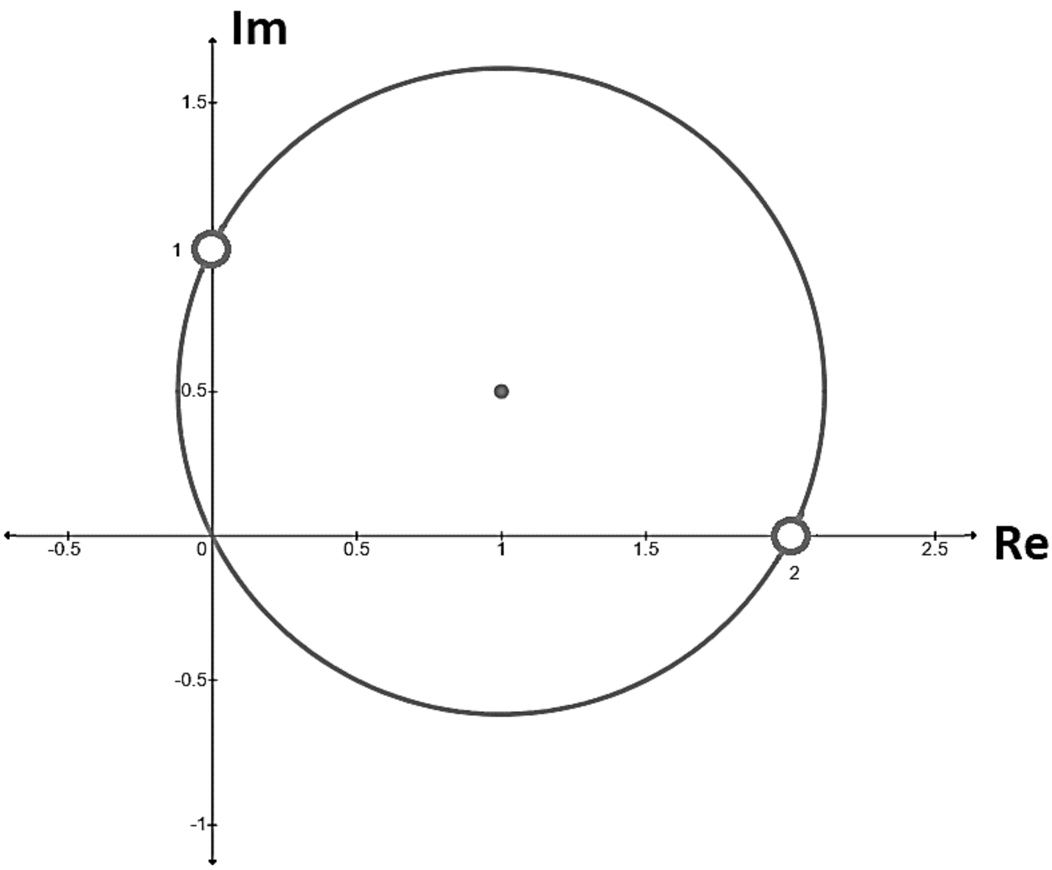
$$x^2 - 2x + y^2 - y = 0$$

$x^2 - 2x + 1 + y^2 - y + \frac{1}{4} = \frac{5}{4}$

$(x-1)^2 + (y - \frac{1}{2})^2 = \frac{5}{4}$

Circle with centre $(1, \frac{1}{2})$ and radius $\frac{\sqrt{5}}{2}$ units.

There are restrictions to the circle, i.e. $w \neq 0$, so $z - 2 \neq 0$ $z \neq 2$ and $z - i \neq 0$, so $z \neq i$



Question 16 (c)(i)

Criteria	Marks
• Correctly mentions the concavity of $y = \ln x$ to make the claim that Trapezoidal rule is an underestimate, hence, $\int_1^n \ln x \; dx > \frac{1}{2}\ln n + \ln(n-1)!$	2
• Uses Trapezoidal rule with $n-1$ subintervals to show that the estimate of the integral is $\int_1^n \ln x \; dx \approx \frac{1}{2}\ln n + \ln(n-1)!$	1

Sample answer:

Applying Trapezoidal rule with $n-1$ sub-intervals.

$$Area \approx \frac{n-1}{2(n-1)}(\ln 1 + \ln n + 2(\ln 2 + \ln 3 + \ln 4 + \cdots + \ln(n-1)))$$

$$\approx \frac{1}{2}(\ln n + 2(\ln(2 \times 3 \times 4 \times \ldots \times (n-1))))$$

$$\approx \frac{1}{2}(\ln n + 2\ln(n-1)!)$$

$$\approx \frac{1}{2}\ln n + \ln(n-1)!$$

Since $y = \ln x$ is concave down, then Trapezoidal rule is an underestimate.

Hence, $\int_1^n \ln x \; dx > \frac{1}{2}\ln n + \ln(n-1)!$

Question 16 (c)(ii)

Criteria	Marks
• Correctly simplifies and rearranges the inequality result to yield the desired result by taking the exponential of both sides.	2
• Uses the result from part i) and substitutes $\int_1^n \ln x \; dx = n\ln n - n + 1$ to yield an inequality result.	1

Sample answer:

Using the result $\int_1^n \ln x \; dx > \frac{1}{2}\ln n + \ln(n-1)!$ From part i)

Let $\int_1^n \ln x \; dx = n\ln n - n + 1$

$$\therefore n\ln n - n + 1 > \frac{1}{2}\ln n + \ln(n-1)!$$

$$e^{n\ln n - n + 1} > e^{\frac{1}{2}\ln n + \ln(n-1)!}$$

$$n^n e^{1-n} > n^{\frac{1}{2}}(n-1)!$$

$$n^{n-\frac{1}{2}} e^{1-n} > (n-1)!$$

$$n \times n^{n-\frac{1}{2}} e^{1-n} > n \times (n-1)!$$

$n^{n+\frac{1}{2}}e^{1-n} > n!$

Hence, $n! < n^{n+\frac{1}{2}}e^{1-n}$

Proof complete.

Question 16 (d)(i)

Criteria	Marks
• Correctly proves the statement using appropriate algebraic working out.	1

Sample answer:

Let $(0,0,0)$ and (a, a, a) be the points which form the longest diagonal in a cube with side length a units.

Length $= \sqrt{a^2 + a^2 + a^2} = \sqrt{3a^2} = \sqrt{3}a$

Proof complete.

Question 16 (d)(ii)

Criteria	Marks
• Correctly proves the statement using appropriate algebraic working out.	1

Sample answer:

The longest diagonal is AG which is also the diameter of the sphere. Hence, $AG = 2$

$2 = \sqrt{3}a$

$$a = \frac{2}{\sqrt{3}}$$

Proof complete.

Question 16 (d)(iii)

Criteria	Marks
• Correctly finds the sum of the volume using appropriate algebraic working out.	1

Sample answer:

6 square pyramids are required to form the entire volume of the cube.

$$\therefore Volume = \frac{2}{6} \times a^3 = \frac{1}{3}\left(\frac{2}{\sqrt{3}}\right)^3 = \frac{8\sqrt{3}}{27}$$

Question 16 (d)(iv)

Criteria	Marks
• Correctly demonstrates the scalar dot product using appropriate algebraic working out.	1

Sample answer:

Point $A(-\frac{1}{2}a, \frac{1}{2}a, \frac{1}{2}a)$ and $C(\frac{1}{2}a, -\frac{1}{2}a, \frac{1}{2}a)$

$$\overrightarrow{OA} \cdot \overrightarrow{OC} = -\frac{1}{4}a^2 - \frac{1}{4}a^2 + \frac{1}{4}a^2 = -\frac{1}{4}a^2 = -\frac{1}{4}\left(\frac{4}{3}\right) = -\frac{1}{3}$$

Proof complete.